UTILITY-BASED RESOURCE ALLOCATION UNDER UNCERTAINTY

DARIUSZ GĄSIOR,[1] KRZYSZTOF BRZOSTOWSKI[1], IGOR PERKO[2]

[1] Wroclaw University of Science and Technology, Faculty of Information and Communication Technology, Wroclaw, Poland
dariusz.gasior@pwr.edu.pl, krzysztof.brzostowski@pwr.edu.pl
[2] University of Maribor, Faculty of Economics and Business, Maribor, Slovenia
igor.perko@um.si

Abstract Due to the COVID-19 pandemic, societies have recently become aware that all decision-making processes are made under huge uncertainty. Since the worldwide situation cannot be compared to any other in the past, it is hard to apply any of the typical descriptions of uncertainty based on historical data. In this paper, the authors try to show how to use expert knowledge of the unknown values of systems parameters to optimise their operation through appropriate allocation of resources and also consider the systems that may be modelled by using the utility theory. Production plants and computer networks are examples of such systems. The authors have modelled the uncertainty with the formalism of uncertain variables and proposed a new approach to the problem of optimising resources with uncertain parameters. A method to solve such a defined problem is also discussed.

Keywords:
uncertainty,
optimisation,
allocation,
production,
manufacturing

University of Maribor Press

DOI https://doi.org/10.18690/um.epf.5.2022.43
ISBN 978-961-286-600-6

1 Introduction

Many systems, such as manufacturing systems, transport systems, information systems and computer networks, need the appropriate allocation of resources in order to operate effectively. To optimise the operation of these systems, it is crucial to know their parameters. Unfortunately, precise values are usually not known in advance and instead some additional descriptions may be available. For example, the probability distribution may be given. However, this is only possible if reliable historical data is available. When faced with a unique situation, one approach is to try to utilise expert knowledge, if available. In this paper, the authors introduce a possible approach for such a case.

2 Deterministic resource allocation problem

2.1 Mathematical model

Let us consider the resource allocation system that transforms resources into goods. Assume that the system consists of J parallel plants that share resources. The system produces I different types of goods. L different resources are available and the amount of the lth resource available is given by C_l. The a_{ijl} transformation coefficient indicates how much of the lth resource is used by the jth plant to produce one unit of the ith good. The income from selling particular goods depends on the utility and the 'willingness-to-pay' coefficient w_i. The utility depends on the amount of goods that are produced and can be modelled with an iso-elastic function $\varphi(\sum_j x_{ij})$. It is also assumed that there are some demands in relation to the minimal amount of each of the goods D_i.

The aim of this paper is to determine $x = [x_{ij}]$, i.e. the amount of each good i that will be produced in each plant j in order to maximise the total income. Furthermore, no resource constraints may be violated and the demand requirements must be fulfilled. This decision results in the appropriate resource allocations, i.e. the amount of the lth resource that is allocated to produce ith good in the jth plant may be calculated as $a_{ijl}x_{ij}$.

2.2 Problem formulation

The problem under consideration may now be formulated as the following optimisation task.
Given: $a_{ijl}, C_l, w_i, \varphi$

Finding:

$$x^* = \arg\max_{x \geq 0} \sum_i w_i \varphi(\sum_j x_{ij}) \quad (1)$$

such that:

$$\sum_i \sum_j a_{ijl} x_{ij} \leq C_l \quad \text{for } l = 1, 2, \ldots, L \quad (2)$$

$$\sum_j x_{ij} \geq D_i \quad \text{for } i = 1, 2, \ldots, I. \quad (3)$$

2.2 Lagrange function and Karush-Kuhn-Tucker conditions

To solve the optimisation problem (1), the Karush-Kuhn-Tucker (KKT) conditions may be considered (for more details see e.g. (Gasior and Orski, 2014)). Thus, the Lagrange function must be introduced:

$$L(x, \lambda, \mu) = -\sum_i w_i \varphi(\sum_j x_{ij}) + \sum_l \lambda_l \left(\sum_i \sum_j a_{ijl} x_{ij} - C_l\right) + \sum_i \mu_i \left(D_i - \sum_j x_{ij}\right). \quad (4)$$

The Karush-Kuhn-Tucker conditions have the following form:

$$\frac{\partial L(x,\lambda,\mu)}{\partial x_{ij}} = 0 \quad \text{for } i = 1, 2, \ldots, I, j = 1, 2, \ldots, J, \quad (5)$$

$$\lambda_l \frac{\partial L(x,\lambda,\mu)}{\partial \lambda_l} = 0 \quad \text{for } l = 1, 2, \ldots, L, \quad (6)$$

$$\frac{\partial L(x,\lambda,\mu)}{\partial \lambda_l} \leq 0 \quad \text{for } l = 1, 2, \ldots, L, \quad (7)$$

$$\lambda_l \frac{\partial L(x,\lambda,\mu)}{\partial \lambda_l} = 0 \quad \text{for } l = 1, 2, \ldots, L, \tag{8}$$

$$\frac{\partial L(x,\lambda,\mu)}{\partial \mu_i} \leq 0 \text{ for } i = 1, 2, \ldots, I, \tag{9}$$

$$x_{ij} \geq 0 \text{ for } i = 1, 2, \ldots, I, j = 1, 2, \ldots, J, \text{ and } \lambda_l \geq 0 \quad \text{for } l = 1, 2, \ldots, L. \tag{10}$$

Therefore, taking into account the Lagrange function form given by (4), (5)-(10) may be rewritten in the following form:

$$-w_i\varphi'(\textstyle\sum_j x_{ij}) + \sum_l \lambda_l \left(\sum_i \sum_j a_{ijl}\right) + \mu_i = 0 \text{ for } i = 1, 2, \ldots, I, j = 1, 2, \ldots, J, \tag{11}$$

$$\lambda_l\left(\textstyle\sum_i \sum_j a_{ijl}x_{ij} - C_l\right) = 0 \text{ for } l = 1, 2, \ldots, L, \tag{12}$$

$$\textstyle\sum_i \sum_j a_{ijl}x_{ij} - C_l \leq 0 \text{ for } l = 1, 2, \ldots, L, \tag{13}$$

$$\mu_i\left(D_i - \textstyle\sum_j x_{ij}\right) = 0 \text{ for } i = 1, 2, \ldots, I, \tag{14}$$

$$D_i - \textstyle\sum_j x_{ij} \leq 0 \text{ for } i = 1, 2, \ldots, I, \tag{15}$$

$$x_{ij} \geq 0 \text{ for } i = 1, 2, \ldots, I, j = 1, 2, \ldots, J, \text{ and } \lambda_l \geq 0 \quad \text{for } l = 1, 2, \ldots, L. \tag{16}$$

The $\varphi(\sum_j x_{ij})$ are so-called iso-elastic utility functions, which means they are continuous, increasing, strictly concave and twice differentiable. Thus, the problem given (1) is a convex optimisation problem, therefore the KKT conditions are necessary and sufficient for optimality. Nevertheless, it is usually difficult to analytically solve the set of equations and inequalities given by (11)-(16), therefore many numerical methods are usually applied.

3 Uncertain variables

To model uncertainty, uncertain variables may be used. This is especially useful when the possible values of the unknown parameters cannot be deduced from the historical data but are based on expert knowledge. A detailed introduction to the formalism of uncertain variables may be found in Bubnicki (2003). The authors of this paper have limited it to only the most important aspects.

The uncertain variables are based on the multi-value logic. Therefore, it is not possible to say whether or not the particular property Ψ, being the logic proposition, is true. Instead, the degree of certainty that the given property is satisfied has been described $v[\Psi]$. For the uncertain variable denoted by $\bar{b}$, two fundamental properties are introduced:

- $\bar{b} \cong b$ which means $\bar{b}$ is approximately equal to *b*
- $\bar{b} \mathrel{\tilde{\in}} D_b$ which means $\bar{b}$ approximately belongs to D_b

The certainty degree $v(\bar{b} \cong b) = h(b)$ is given by an expert for every value *b* and *h*(*b*) is called certainty distribution. Introducing uncertain variables, the following definitions must be also given:

$$v\left(\bar{b} \mathrel{\tilde{\in}} D_b\right) = \begin{cases} \max\limits_{b \in D_b} h(b) & for \quad D_b \neq \emptyset, \\ 0 & for \quad D_b = \emptyset, \end{cases}$$

$$v\left(\bar{b} \mathrel{\tilde{\notin}} D_b\right) = 1 - v\left(\bar{b} \mathrel{\tilde{\in}} D_b\right),$$

$$v\left(\bar{b} \mathrel{\tilde{\in}} D_1 \vee \bar{b} \mathrel{\tilde{\in}} D_2\right) = \max\{v\left(\bar{b} \mathrel{\tilde{\in}} D_1\right), v\left(\bar{b} \mathrel{\tilde{\in}} D_2\right)\},$$

$$v\left(\bar{b} \mathrel{\tilde{\in}} D_1 \vee \bar{b} \mathrel{\tilde{\in}} D_2\right) = \begin{cases} \min\left\{v\left(\bar{b} \mathrel{\tilde{\in}} D_1\right), v\left(\bar{b} \mathrel{\tilde{\in}} D_2\right)\right\} & \text{for} \quad D_1 \cup D_2 \neq \emptyset, \\ 0 & \text{for} \quad D_1 \cup D_2 = \emptyset. \end{cases}$$

4 Resource allocation problem for an uncertain case

It must be stressed that the precise values of some parameters may not be known in advance.

For instance, how much customers are willing to pay for the achieved utility may not be known. The number of available resources may also not be known in advance, since it is not always possible to ascertain whether all the supplies of resources will be on time or if the full capacity of the devices required for production will be available. Finally, the precise values of resources that are need to produce one unit of a good may not be known. It is sometimes possible to predict these values based on historical data. However, such data may not be available for cases covering incidents such as the COVID-19 pandemic or war, which may have a huge impact on the plant under consideration. In such cases, expert opinion is the only reliable option. Below, an approach for such a case is described.

Let us now consider the most general case, for which a_{ijl}, C_l, w_i are the uncertain parameters and there is an expert who describes their knowledge of the possible values of these parameters in terms of the certainty distributions h_{aijl}, h_{Cl}, h_{wi}, respectively. If any of the parameters (i.e. p) value (p^*)) are precisely known in advance, the following distribution may be assumed: $h(p) = 1(0)$ for $p = p^*$ (otherwise) without loss of generality.

The classic approach cannot now be directly applied. Thus, the optimisation problem is usually formulated differently. The first option is to maximise the certainty index so that the constraints are fulfilled and the objective function must not be less than the given threshold α. The other option consists of finding a solution for which the threshold α is maximal and the certainty index of fulfilling the appropriate requirements is not less than the given minimal acceptable value v^* (Gasior, 2008). Note that in both cases, additional user input is required.

In this paper, the authors propose a different approach to the optimisation problem by introducing the certainty index that the KKT conditions are approximately satisfied, which is denoted as follows:

$$v\left(\left(\forall_{ij} - w_i\varphi'(\sum_j x_{ij}) + \sum_l \lambda_l \left(\sum_i \sum_j a_{ijl}\right) + \mu_i \cong 0\right)\right.$$

$$\wedge\left(\forall_l \lambda_l \left(\sum_i \sum_j \bar{a}_{ijl} x_{ij} - \bar{C}_l\right) \cong 0\right) \wedge \left(\forall_l \sum_i \sum_j \bar{a}_{ijl} x_{ij} - \bar{C}_l \lesssim 0\right)$$

$$\left.\wedge\left(\forall_i \ \mu_i \left(D_I - \sum_j x_{ij}\right) = 0\right) \wedge \left(\forall_i \ D_i - \sum_j x_{ij} \leq 0\right)\right) \triangleq v(x, \lambda, \mu),$$

and by then maximising it for non-negative x_{ij} and λ_l, i.e.:

$$x^*, \lambda^* = \arg\max_{x \geq 0, \lambda \geq 0} v(x, \lambda, \mu)\,.$$

Let us introduce an auxiliary notation:

$$v\left(-w_i\varphi'(\sum_j x_{ij}) + \sum_l \lambda_l \left(\sum_i \sum_j a_{ijl}\right) + \mu_i \cong 0\right) \triangleq v_{ij}^{(1)}(x, \lambda, \mu),$$

$$v\left(\lambda_l \left(\sum_i \sum_j \bar{a}_{ijl} x_{ij} - \bar{C}_l\right) \cong 0\right) \triangleq v_l^{(2)}(x, \lambda, \mu),$$

$$v\left(\sum_i \sum_j \bar{a}_{ijl} x_{ij} - \bar{C}_l \lesssim 0\right) \triangleq v_l^{(3)}(x, \lambda, \mu).$$

It is also notable that:

$$\left(\forall_i \ \mu_i\left(D_i - \textstyle\sum_j x_{ij}\right) = 0\right) \wedge \left(\forall_i \ D_i - \textstyle\sum_j x_{ij} \leq 0\right) \quad (17)$$

may be either true or false. Therefore, the certainty index $v\Big(\big(\forall_i\ \mu_i\big(D_I - \sum_j a_{ijl}x_{ij}\big) = 0\big) \wedge \big(\forall_i\ D_i - \sum_j a_{ijl}x_{ij} \le 0\big)\Big)$ may be either 1 or 0 respectively. Any solution is valid for the certainty index equal to 0, therefore in this paper, the authors are only interested in cases where the certainty index is greater than 0, hence (17) are considered as constraints.

The problem formulation may now be rewritten as:

$$x^*, \lambda^* = \arg\max_{x\ge 0,\lambda\ge 0,\mu\ge 0} \min\{\min_{ij} v_{ij}^{(1)}(x,\lambda,\mu), \min_{l} v_l^{(2)}(x,\lambda,\mu), \min_{l} v_l^{(3)}(x,\lambda,\mu)\} \tag{18}$$

such that constraints (14) and (15) are satisfied.

Instead of solving the problem (18), an equivalent problem may be formulated.

Finding:

$$x^*, \lambda^*, \mu^*, v^* = \arg\max_{x\ge 0,\lambda\ge 0,\mu\ge 0,v^*\in[0,1]} v^*$$

such that:

$v_{ij}^{(1)}(x,\lambda,\mu) \ge v^*$, $\forall_1 v_l^{(2)}(x,\lambda,\mu) \ge v^*$, $\forall_1 v_l^{(3)}(x,\lambda,\mu) \ge v^*$ for $i = 1, 2, \ldots, I$, $j = 1, 2, \ldots, J$,

$\mu_i\big(D_i - \sum_j a_{ijl}x_{ij}\big) = 0$ for $i = 1, 2, \ldots, I$, and $D_i - \sum_j x_{ij} \le 0$ for $i = 1, 2, \ldots, I$

According to the properties of the uncertain variables, the introduced certainty indices may be determined as follows:

$$v_{ij}^{(1)}(x,\lambda,\mu) = \max_{w_i, a_{ijl}:\ -w_i\varphi'(\sum_j x_{ij}) + \sum_l \lambda_l(\sum_i \sum_j a_{ijl}) + \mu_i = 0} \min\{h_{wi}(w_i), \min_{ijl} h_{aijl}\big(a_{ijl}\big)\}, \tag{19}$$

$$v_l^{(2)}(x,\lambda,\mu) = \max_{C_l, a_{ijl}:\, \lambda_l(\sum_i \sum_j a_{ijl}x_{ij} - C_l) = 0} \min\{h_{Cl}(C_l), \min_{ijl} h_{aijl}(a_{ijl})\}\,, \tag{20}$$

$$v_l^{(3)}(x,\lambda,\mu) = \max_{C_l, a_{ijl}:\, \sum_i \sum_j a_{ijl}x_{ij} - C_l \le 0} \min\{h_{Cl}(C_l), \min_{ijl} h_{aijl}(a_{ijl})\}\,. \tag{21}$$

The following procedure may be applied to solve the formulated problem:

1. When trying to solve the deterministic version of the problem (1) for the values of unknown parameters, expert opinion is the most certain. If this solution is available, it is also the solution to the uncertain problem (18). Thus, the procedure stops.
2. Determine (19), (20), (21) using e.g. the method given in Gasior (2008).
3. Substitute the certainty index in (18) and solve the optimisation problem using the methods given in e.g. Boyd et al (2004).

5 Numerical example

Let us consider the following simple numerical example based on the assumption that there is a company with one factory ($J = 1$) that produces two types of products ($I = 2$), and there is one crucial resource type ($L = 1$). The factory consumes one unit of the resource to produce one unit of each product type ($a_{111} = 1$, $a_{211} = 1$). Due to some agreements, the company must produce at least 50 units of the first product and at least 60 units of the second product ($D_1 = 50, D_2 = 60$). The company may sell more products of each type and its income may be calculated using the following function: $w_1\varphi(x_{11}) + w_2\varphi(x_{21})$ where x_{11} and x_{21} are the amount of the first and the second product respectively. The precise value of the available resource (C_1) and 'willingness-to-pay' coefficients (w_1, w_2) are not known in advance. Instead, the possible values of these parameters are given by the expert in terms of the triangular certainty distributions h_{C1}, h_{w1}, h_{w2} with the following parameters: $C_1^* = 100, d_{C1} = 20, w_1^* = 10, d_{w1} = 1, w_2^* = 20, d_{w2} = 2$

(compare e.g. (Gasior, 2008)). The aim here is to determine x_{11} and x_{21},which solves (18).

Solution:

1. Note that even for minimal demands (i.e. for $x_{11} = 50,$ and $x_{21} = 60$), there is no feasible solution for the deterministic version of the problem $for\ C = C_1^* = 100, w_1 = w_1^* = 10, w_2 = w_2^* = 20$. In such a case, the resource constraint (2) is not satisfied, since $50 + 60 - 100 > 0$. Thus, the procedure must be continued.
2. The certainty indices (19), (20), (21) have the following forms: $v_{i1}^{(1)}(x, \lambda) = h_{wi}\big((\lambda_1 + \mu_i)x_{i1}\big)$, $v_l^{(2)}(x, \lambda) = h_{C1}(x_{11} + x_{21})$, and $v_l^{(3)}(x, \lambda) = \frac{C_1^* - (x_{11} + x_{21})}{d_{C1}}$ respectively.
3. Therefore, the optimal solution to the problem (18) is: $x_{11} = 50$, $x_{21} = 60$, and $v^* = 0.5$.

6 Conclusions

In this paper, the authors considered the problem of resource optimisation under uncertainty. They showed how to model the uncertainty using the formalism of uncertain variables and proposed an approach consisting of a maximisation certainty index that approximate fulfils the necessary and sufficient conditions of optimality (KKT) for the formulated problem of resource allocation problem and also discuss how it should be solved.

References

Boyd, S., Boyd, S. P., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.

Bubnicki, Z. (2003). *Analysis and decision making in uncertain systems*. Springer Science & Business Media.

Gąsior, D. (2008). QoS rate allocation in computer networks under uncertainty. *Kybernetes*. 37(5)

Gąsior, D., & Orski, D. (2014). On the rate allocation problem under co-existence of uncertain utility function parameters and uncertain link capacities. *Journal of the Operational Research Society*, *65*(10), 1562-1570.