

Fast Recognition of Some Parametric Graph Families

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ABSTRACT

Recognizing graphs with high level of symmetries is hard in general, and usually requires additional structural understanding. In this paper we study a particular graph parameter and motivate its usage by devising efficient recognition algorithm for the family of I -graphs.

For integers ℓ, λ, m a simple graph is $[\ell, \lambda, m]$ -cycle regular if every path of length ℓ belongs to exactly λ different cycles of length m . We identify all $[1, \lambda, 8]$ -cycle regular I -graphs and, as a consequence, describe linear recognition algorithm for the observed family.

Similar procedure can be used to devise the recognition algorithms for Double generalized Petersen graphs and folded cubes. Besides that, we believe the structural observations and methods used in the paper are of independent interest and could be used for solving other algorithmic problems.

Keywords I -graphs, double generalized Petersen graphs, folded cubes, recognition algorithm, cycle regularity.

1 Introduction

Important graph classes such as bipartite graphs, (weakly) chordal graphs, perfect graphs and forests are defined or characterized by their cycle structure. A particularly strong description of a cyclic structure is the notion of *cycle-regularity*, introduced by Mollard [11]:

For integers l, λ, m a simple graph is $[l, \lambda, m]$ -cycle regular if every path on $l + 1$ vertices belongs to exactly λ different cycles of length m .

It is perhaps natural that cycle-regularity mostly appears in the literature in the context of symmetric graph families such as hypercubes, Cayley graphs or circulants.

Understanding the structure of subgraphs of hypercubes which avoid all 4-cycles does not seem to be easy. Indeed, a question of Erdős regarding how many edges can such a graph contain remains open after more than 30 years [5].

In this paper we study cycle-regularity and more general cyclic aspects of a family of I -graphs, with the focus of devising an efficient recognition algorithm. Similar approach can be extended also to two other graph families,

namely Double generalized Petersen graphs and folded cubes. Due to the space constraints the study of these two families is not covered here, therefore we defer interested readers to the full version of this paper [9].

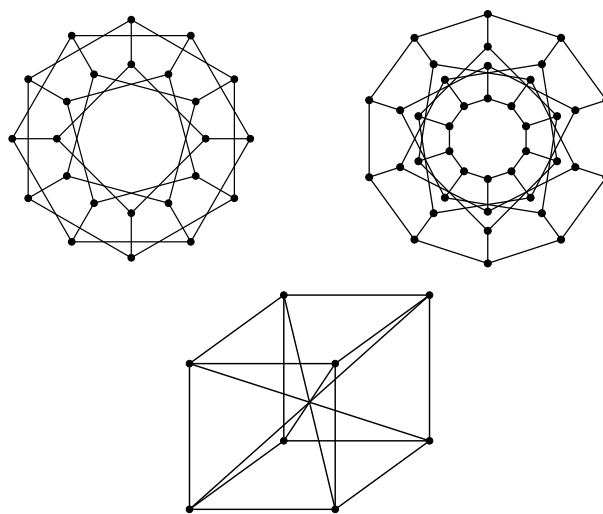


Figure 1: I -graph $I(12, 2, 3)$, double generalized Petersen graph $DP(10, 2)$, and folded cube FQ_4 .

I -graphs were introduced in the Foster census [6], and are trivalent or cubic graphs with edges connecting vertices of two star polygons. They form a natural generalization of the well-known *generalized Petersen graphs* introduced in 1950 by Coxeter [3] and later named by Watkins in 1969 [14]. The family of I -graphs has been studied extensively with respect to their automorphism group and isomorphisms [1, 7, 13], Hamiltonicity [2], spectrum [12], and independence number [4, 8].

Our first result identifies all $[1, \lambda, 8]$ -cycle regular members and determines the corresponding values of λ .

Theorem 1. *An arbitrary I -graph is never $[1, \lambda, 8]$ -cycle regular, except when isomorphic to $I(n, j, k)$ where $j = 1$ and $(n, k) \in \{(3, 1), (4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (13, 5), (24, 5), (26, 5)\}$.*

These structural results are used to devise the recognition algorithm for I -graphs.

Theorem 2. *I -graphs can be recognized in linear time.*

If the input graph is a member of the observed family, we

not only provide its parameters but also give a certificate of correctness, i.e. we give an exact isomorphism.

1.1 Preliminaries

Unless specified otherwise, all graphs in this paper are finite, simple, undirected and connected. For a given graph G we use a standard notation for a set of vertices $V(G)$ and a set of edges $E(G)$. A k -cycle C in G , on vertices v_1, v_2, \dots, v_k from $V(G)$ is denoted as (v_1, \dots, v_k) . For integers a and b we denote with $\gcd(a, b)$ the greatest common divisor of a and b respectively.

Definition 3. Let l, λ, m be positive integers. A simple graph G is $[l, \lambda, m]$ -cycle regular if every path on $l + 1$ vertices of G belongs to exactly λ different m -cycles of G .

It is easy to see that $[1, \lambda, 8]$ -cycle regular cubic graphs are also $[0, 3\lambda/2, 8]$ -cycle regular, but the converse does not hold. Related to this we define a function $\sigma : E(G) \mapsto \mathbb{N}$, where $\sigma(e)$ corresponds to the number of distinct 8-cycles an edge e belongs to. We call $\sigma(e)$ an *octagon value* of an edge e , and we say that a graph G has a *constant octagon value* if σ is a constant function.

2 Structural analysis

Before we start with the analysis of the observed graph family we need to formally define it and present some of its basic properties.

Definition 4. Let n, j, k be positive integers for which $n \geq 3$ and $n \geq j, k \geq 1$. An I -graph $I(n, j, k)$ is a graph on vertices $\{u_0, u_1, \dots, u_{n-1}, w_0, w_1, \dots, w_{n-1}\}$, with the edge set consisting of outer edges $u_i u_{i+j}$, inner edges $w_i w_{i+k}$ and spoke edges $u_i w_i$, where the subscripts are taken modulo n .

Without loss of generality we always assume that $j, k < n/2$. Since $I(n, j, k)$ is isomorphic to $I(n, k, j)$, we restrict ourselves to cases when $j \leq k$. It is well known [1] that an I -graph $I(n, j, k)$ is disconnected whenever $d = \gcd(n, j, k) > 1$. In this case it consists of d copies of $I(n/d, j/d, k/d)$. Therefore, throughout the paper we consider only graphs $I(n, j, k)$ where $\gcd(n, j, k) = 1$. We also know [7] that two I -graphs $I(n, j, k)$ and $I(n, j', k')$ are isomorphic if and only if there exists an integer a , which is relatively prime to n , for which either $\{j', k'\} = \{aj \pmod{n}, ak \pmod{n}\}$ or $\{j', k'\} = \{aj \pmod{n}, -ak \pmod{n}\}$. Throughout the paper, whenever we discuss I -graphs with certain parameters, we consider only the lexicographically smallest possible parameters by which the graph is uniquely determined.

2.1 Equivalent 8-cycles

A particular member of automorphism group of every I -graph is a *rotation* defined as: $\rho(u_i) = u_{i+1}$, $\rho(w_i) = w_{i+1}$. Clearly, applying n times the rotation ρ yields an identity automorphism. When acting on I -graphs with ρ we get 3 edge orbits: orbit of outer edges E_J , orbit of spoke edges E_S and orbit of inner edges E_I . Edges from the same orbit E_J, E_S , or E_I have the same octagon value, which we denote by σ_J, σ_S and σ_I , respectively.

Therefore the *octagon value of an I -graph* is said to be a triple $(\sigma_J, \sigma_S, \sigma_I)$.

We say that two 8-cycles of an I -graph are *equivalent* if we can map one into the other using rotation ρ . Let $G \simeq I(n, j, k)$ be an arbitrary I -graph and let C be one of its 8-cycles. With $\gamma(C)$ we denote the number of equivalent 8-cycles to C in G . Each 8-cycle contributes to the octagon value of an I -graph. We denote the contributed amount with $\tau(C)$, defined as the triple $(\delta_j, \delta_s, \delta_i)$, where we calculate $\delta_j, \delta_s, \delta_i$ by counting the number of outer, spoke and inner edges of a cycle and multiply these numbers with $\gamma(C)/n$. If a graph G admits m non-equivalent 8-cycles C_1, C_2, \dots, C_m , one may calculate its octagon value $(\sigma_J, \sigma_S, \sigma_I)$ as $\sum_{i=1}^m \tau(C_i)$.

The following claim serves also as an example of the above-mentioned definitions.

Claim 5. For $I(n, j, k)$ where $n > 3$ and integers $k, j < n/2$ there always exists an 8-cycle.

Indeed, if $k \neq j$ it is of the form

$$C^* = (w_0, w_{\pm k}, u_{\pm k}, u_{\pm k \pm j}, w_{\pm k \pm j}, w_{\pm j}, u_{\pm j}, u_0).$$

If $k = j$ it is of the form

$$C_7 = (u_0, u_k, u_{2k}, u_{3k}, w_{3k}, w_{2k}, w_k, w_0).$$

2.2 Characterization of non-equivalent 8-cycles

Our aim is to identify all possible 8-cycles that can appear in an arbitrary I -graph and determine their contribution towards the octagon value of the graph. It is easy to see that an arbitrary 8-cycle can have either 4, 0 or 2 spoke edges, so we obtained this list by distinguishing 8-cycles by the number of spoke edges they admit. In the case of the 8-cycle admitting 2 spoke edges, we further distinguish cases by the number of outer and inner edges within a given 8-cycle.

Due to space constraints we present the analysis of 8-cycles admitting 4 spoke edges only, as it is the easiest case to deal with (for remaining cases see [9]). This analysis leads to complete characterisation of 8-cycles for the family of I -graphs, presented in the Table 1.

8-cycles with 4 spoke edges In addition to 4 spoke edges the 8-cycle must also have two inner and two outer edges. When using the spoke edge there are two options for choosing an inner (outer) edge. After considering all cases it is easy to see that there can be just two such 8-cycles, C^* (see Claim 5), which exists whenever $j \neq k$, and C_0 , which is of the following form:

$$C_0 = (w_0, w_{\pm k}, u_{\pm k}, u_{\pm k \pm j}, w_{\pm k \pm j}, w_{\pm 2k \pm j}, u_{\pm 2k \pm j}, u_{\pm 2k \pm 2j}).$$

Cycle C_0 exists whenever $2k + 2j = n$. One can verify easily, that n applications of the rotation ρ to C^* and $n/2$ applications of the rotation ρ to cycle C_0 maps the cycle back to itself. Therefore there are n equivalent cycles to C^* and $n/2$ equivalent cycles to C_0 in an I -graph $I(n, j, k)$ and they contribute $(2, 4, 2)$ and $(1, 2, 1)$, respectively, to the graph octagon value.

2.3 Obtaining constant octagon value

Every 8-cycle of an I -graph contributes to the octagon value of each edge partition. It turns out that if we can



Label	A representative of an 8-cycle	Existence conditions	$\tau(C)$	$\gamma(C)$
C^*	$(w_0, w_{\pm k}, u_{\pm k}, u_{\pm k \pm j}, w_{\pm k \pm j}, w_{\pm j}, u_{\pm j}, u_0)$	$k \neq j$ and $n > 4$	$(2, 4, 2)$	n
C_0	$(w_0, w_{\pm k}, u_{\pm k}, u_{\pm k \pm j}, w_{\pm k \pm j}, w_{\pm 2k \pm j}, u_{\pm 2k \pm j}, u_{\pm 2k \pm 2j})$	$2k + 2j = n$	$(1, 2, 1)$	$n/2$
C_1	$(u_0, u_j, u_{2j}, u_{3j}, u_{4j}, u_{5j}, u_{6j}, u_{7j})$	$8j = n$ or $3n$	$(0, 0, 1)$	$n/8$
C_2	$(w_0, w_k, w_{2k}, w_{3k}, w_{4k}, w_{5k}, w_{6k}, w_{7k})$	$8k = n$ or $3n$	$(1, 0, 0)$	$n/8$
C_3	$(w_0, w_k, w_{2k}, w_{3k}, w_{4k}, w_{5k}, u_{5k}, u_{5k+j})$	$5k + j = n$ or $2n$	$(5, 2, 1)$	n
	$(w_0, w_k, w_{2k}, w_{3k}, w_{4k}, w_{5k}, u_{5k}, u_{5k-j})$	$5k - j = n$ or $2n$		
C_4	$(u_0, u_j, u_{2j}, u_{3j}, u_{4j}, u_{5j}, w_{5j}, w_{5j+k})$	$k + 5j = n$ or $2n$	$(1, 2, 5)$	n
	$(u_0, u_j, u_{2j}, u_{3j}, u_{4j}, u_{5j}, w_{5j}, w_{5j-k})$	$5j - k = 2n$ or n or 0		
C_5	$(w_0, w_k, w_{2k}, w_{3k}, w_{4k}, u_{4k}, u_{4k+j}, u_{4k+2j})$	$4k + 2j = n$ or $2k + j = n$	$(4, 2, 2)$	n
	$(w_0, w_k, w_{2k}, w_{3k}, w_{4k}, u_{4k}, u_{4k-j}, u_{4k-2j})$	$4k - 2j = n$		
C_6	$(u_0, u_j, u_{2j}, u_{3j}, u_{4j}, w_{4j}, w_{4j+k}, w_{4j+2k})$	$2k + 4j = n$ or $k + 2j = n$	$(2, 2, 4)$	n
	$(u_0, u_j, u_{2j}, u_{3j}, u_{4j}, w_{4j}, w_{4j-k}, w_{4j-2k})$	$4j - 2k = n$ or 0		
C_7	$(w_0, w_k, w_{2k}, w_{3k}, u_{3k}, u_{3k+j}, u_{3k+2j}, u_{3k+3j})$	$3k + 3j = n$ or $2n$	$(3, 2, 3)$	n
	$(w_0, w_k, w_{2k}, w_{3k}, u_{3k}, u_{3k-j}, u_{3k-2j}, u_{3k-3j})$	$3k - 3j = n$ or 0		

Table 1: All non-equivalent 8-cycles of I -graphs, their existence conditions, their contribution towards the octagon value of an I -graph τ , number of their equivalent cycles in an I -graph γ .

identify at least one edge partition of a graph, we can easily determine its parameters. Therefore, we want to find graphs with constant octagon value. These are graphs for which all edges touch the same number of 8-cycles. They are called $[1, \lambda, 8]$ -cycle regular graphs. We consider all possible collections of 8-cycles and determine octagon values of I -graphs admitting those 8-cycles. Since I -graphs are defined with 3 parameters and all 8-cycles give constraints for these parameters, it is enough to consider collections of at most 4 cycles, to uniquely determine all $[1, \lambda, 8]$ -cycle regular graphs. After a thorough analysis (see [9]) we see that the list of $[1, \lambda, 8]$ -cycle regular I -graphs consists of 10 members (see Figure 2). Surprisingly, it turns out that all $[1, \lambda, 8]$ -cycle regular I -graphs are in the family of generalized Petersen graphs. This proves Theorem 1.

3 Recognition algorithm

The recognition algorithm relies on the fact that there is just a small number of I -graphs (ten) with the constant octagon value. In particular, whenever the input graph G of the Algorithm 1 is a member of the family of I -graphs and is not $[1, \lambda, 8]$ -cycle regular, we can immediately identify one of its edge orbits (E_I, E_J , or E_S), of size $|V(G)|/2$. Since the octagon value of each edge is computed in constant time and there is a finite number of the $[1, \lambda, 8]$ -cycle regular I -graphs the Theorem 2 holds.

Correctness and time complexity of the algorithm. We first note, that if G is not cubic then it does not belong to observed graph families. Since checking whether a graph is cubic takes linear time we simply assume that the input graph is cubic. Furthermore, if G is not connected then it can only be a member of the family of I -graphs whenever it consists of multiple copies of a smaller I -graph G' . However, this case can easily be resolved by separately checking each part, so we can assume that the input graph is connected. Algorithm 1 consists of the following 3 parts.

1. Partitioning the edges with respect to the octagon value.

The algorithm determines the octagon value of each edge $e \in E(G)$ and builds a partition set \mathcal{P} of graph

Data: connected cubic graph G
Result: parameters (n, j, k) and isomorphism to $I(n, j, k)$, if it exists.

```

1  $\mathcal{P} \leftarrow$  an empty dictionary
2 for  $e \in E(G)$  do
3    $s = \text{OCTAGONVALUE}(e)$ ;          /* calculate
    $\sigma(e)$  */
4    $\mathcal{P}[s].\text{append}(e)$ 
5 end
6  $U \leftarrow$  an item of  $\mathcal{P}$  with minimum positive
   cardinality
7 if  $G[U]$  is a 2-factor;          /* 2-factor is a
   2-regular graph */
8 then
9    $U \leftarrow \{e \mid e \in E(G), e \text{ is adjacent to an edge}
   \text{ of } U\}$  /* a perfect matching in  $G$  */
10 end
11 return  $\text{EXTEND}(G, U)$ 

```

Algorithm 1: Recognition procedure for I -graphs

edges (see lines 1 – 4). Since G is cubic and all 8-cycles containing edge e consist of edges which are at distance at most 4 from e , it is enough to check a subgraph H of G of order at most 62, to calculate the octagon value of an edge e . Therefore, calculation of $\text{OCTAGONVALUE}(e)$ takes $O(1)$ time for each edge e and this whole part is performed in $\Theta(|E(G)|)$ time.

2. Identifying the edge-orbit which corresponds to the set of spokes.

Throughout lines 6 – 9 we determine the edge-orbit which corresponds to the set of spokes. It is easy to see that this requires additional $O(|E(G)|/3)$ time.

3. Using set U for determining parameters of a given graph.

The algorithm uses computed set U to determine exact isomorphism between G and an I -graph or a double generalized Petersen graph, if it exists. This procedure differentiates regarding the graph family we are considering. The related procedure $\text{EXTEND}(G, U)$ is performed in $\Theta(|E(G)|)$ time.

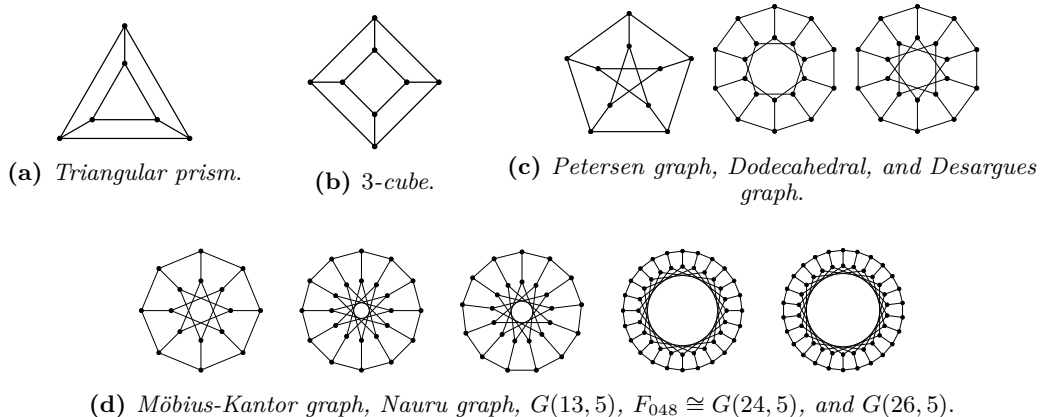


Figure 2: All $[1, \lambda, 8]$ -cycle regular I -graphs. That is, (a) the graph on no 8-cycles; (b) the graph containing C_0 and C_7 ; (c) graphs containing C_5, C_6 and C^* ; and (d) graphs containing C_3, C_4 and C^* .

3.1 Subprocedure $\text{Extend}(G, U)$

It is easy to see that $\text{Extend}(G, U)$ can safely reject G if $|V(G)|$ is not divisible by 2. For this subprocedure set $n = |V(G)|/2$ and denote by H the subgraph of G induced by the vertices of U . There are two possibilities.

$H = G$. In this case graph G has a constant octagon value. Since there are just ten such I -graphs checking G against them takes constant time.

H is of order $2n$ and is 1-regular. Since U is a perfect matching of G the set $E(G) \setminus U$ is a collection of cycles. If G is an I -graph, then there exist positive integers i, j, l_1, l_2 with $j \leq i$ such that there are j cycles of length l_1 and i cycles of length l_2 . It remains to determine parameter k and check whether $G \simeq I(n, j, k)$. This procedure depends on the structure of I -graphs and the 8-cycle C^* . It is performed in $\Theta(|E(G)|)$ time (see [9] for details).

4 Conclusion

Studying the cyclic structure as described in this paper led to the construction of fast recognition algorithms for three parametric families. To the best of our knowledge, in addition to this work, such a procedure was so far only used in [10] for the family of generalized Petersen graphs. We believe that a similar approach should give interesting results for other parametric graph families of bounded degree, such as Johnson graphs, rose window graphs, Tabačjn graphs, Y -graphs, or H -graphs.

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