Do Alternative Algorithms for Two-Digit Multiplication Really Help Students to be More Efficient?

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Abstract Standard arithmetic algorithms are the traditional part of school mathematics. The teaching and the learning of algorithms have been associated with procedures and erroneously with low-level cognition. Teaching algorithms without developing a conceptual understanding is a major concern for many mathematics teachers. However, if efficient teaching and learning strategies are used, this is not necessarily the case. Multi-digit number multiplication proved to be a difficult topic for many young students; therefore, many errors have been reported in the literature. Our research problem was to compare the efficiency of standard algorithms with the efficiency of several alternative two-digit multiplication algorithms. We designed a pedagogical experiment, after which the multiplication fluency of 5th-grade students (N = 73) was measured. Multiplication fluency was measured in two dimensions: Correctness (of the result) and time efficiency. The results show that the introduction of alternative algorithms has not hindered correctness, but the use of alternative algorithms has greatly increased the computing time. On the other hand, the results show that students consistently chose alternative algorithms, or more precisely, area algorithms. On the basis of the results obtained, some guidelines for school practice are given.

Keywords: mathematics education, elementary school, distributive property, two-digit multiplication, alternative algorithms.

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Introduction

Mathematics is defined in the syllabus (Žakelj et al., 2011) as one of the basic subjects in elementary school with numerous educational–informative, functional–formative and educational tasks. Knowledge of certain procedures, understanding, cross-curricular integration, use of mathematical knowledge and ability to solve problems are important factors for living a quality life. In the process of teaching mathematics, among other things, we raise awareness of the practical usefulness and meaningfulness of learning mathematics. The factors of quality education in elementary schools based on the principles of social responsibility are the basis for the progress of the whole society. According to *UNESCO IBE* (International Bureau of Education, n. d) raising the mathematical competence of future citizens has a positive influence on raising the GDP of the entire society.

In the initial years of elementary school, students are introduced to several types of algorithms. The most typical are the algorithms in arithmetic, which we will focus on later. Students also learn about algebraic and geometric algorithms. An example of content where we introduce algebraic algorithms are equations. In Slovenian elementary schools, the pupils are introduced to the process of solving equations in fourth grade. The first geometric constructions with the characteristics of the algorithms are also introduced to Slovenian students in fourth grade when constructing rectangles and squares.

When we talk about multiplication algorithms, we follow the definition by Jazby and Pearn, which states that multiplication algorithms are "cognitive aids that make it possible to break down a multiplication problem into a series of less cognitively demanding subroutines" (2015, p. 311). Multi-digit number multiplication is one of the more difficult concepts in early mathematics, so many of the mistakes that students make are known (for example, see Leung, 2006). On the other hand, there are few studies that deal with teaching strategies for multi-digit multiplication (Larsson, 2016). According to Fuson (2003), it is important that students have the right physical condition and a suitable learning environment in which they can successfully develop the predispositions necessary for understanding the algorithms. In addition, the algorithms must be clearly presented for an independent and successful application based on a conceptual understanding (Fan & Bokhove, 2014).

Multiplication

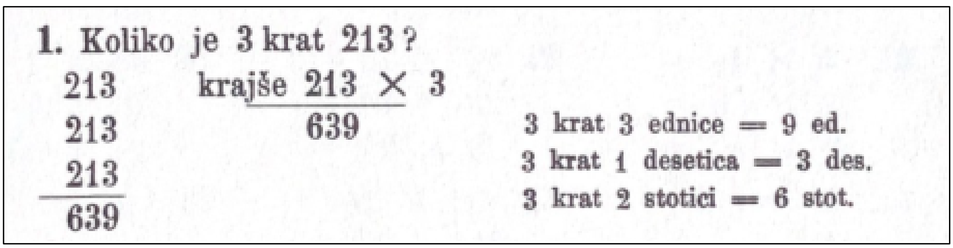
Multiplication is at the core of elementary arithmetic instruction and underpins other mathematical topics such as fractions, ratio, proportionality, and functions (Bakker et al., 2014). The importance of multiplication and division understanding is evident in the National Council of Teachers of Mathematics (NCTM) *Curriculum Focal Points* developed in the US (NCTM, n.d.). Multiplicative reasoning is emphasized as one of the three crucial mathematics themes (along with equivalence and computational fluency) that are interwoven through the content standards for middle grades, forming the foundation for proportional reasoning. Over-emphasizing memorization of facts or developing conceptual understanding along with factual and procedural knowledge is a long-standing problem in mathematics education (Smith, & Smith, 2006). Multi-digit multiplication can be performed through learned algorithms or student-invented strategies. Calculations concerning multiplication and division, whether learned algorithms or student-invented strategies have attracted less research when compared to addition and subtraction (Larsson, 2016). In the early 1980s, most research on calculations tended to focus on conceptual errors in algorithms, while a decade later student-invented strategies became the focus. The latter focus on student-invented strategies has prompted a number of case studies in which students have been engaged in devising methods for calculation, especially concerning multi-digit addition and subtraction (Larsson, 2016).

Multiplication can be represented in various ways, although there is less of a consensus with regard to categorizing multiplicative situations in comparison to additive situations (Fuson, 2003). Some of these representations are contextual, as in word problems or real-life problems. Multiplicative representations frequently found in the literature, include, among other situations, equal groups and rectangular arrays (Greer, 1992). In asymmetrical situations such as equal groups, the multiplier has a different role from the multiplicand. Symmetrical situations, such as rectangular arrays and area, where the two factors have the same role, are more convenient for the development of algorithms. Such multiplicative representations are not simply contextual or visual cues, they can also be perceived as the thinking tool students use when determining what actions to take with regard to the numbers in a problem or explaining properties (Yackel, 2001).

Standard Multiplication Algorithm

The place value system is the foundation of our numbering system. The efficiency of the arithmetic algorithms is based on it. A real understanding of the basic four algorithms rests on a firm grasp of the place value system. Multiplication, for example, is a little more than the combination of the place value system, distributivity, and single-digit math facts for multiplication. This combination is the mathematical reasoning that makes the multiplication algorithm work.

Students in Slovenian elementary schools are taught the standard algorithm for multiplication with one-digit numbers in fourth grade (age 9–10). The algorithm has been first presented in Slovenia in the first Slovenian schoolbook for mathematics, written by Franc Močnik and published in 1856 (Močnik, 1914).


1. Koliko je 3 krat 213?

213 krajše 213 × 3
213 639
213
639

3 krat 3 ednice = 9 ed.
3 krat 1 desetica = 3 des.
3 krat 2 stotici = 6 stot.

Figure 1: Močnik textbook from 1856,
reprinted for Močnik, 1914.

Learned algorithms are often referred to as vertical or standard algorithms. They typically build on the distributive property where both factors are split into ones, tens, hundreds, etc. and each part is multiplied by each part of the other factor. Implicit use of distributivity has been found to develop without instructions in elementary classrooms, focussed on student-invented strategies for multiplication. This was in contrast to commutativity, which was harder for students to discover by themselves (Ambrose et al., 2003). In fifth grade, students are introduced to the multidigit multiplication algorithm (see Figure 2).

	hundreds	tens	ones		tens	ones
	2 2	$3_{5\ 4}$	8	·	7	6
	2	6	6	0		
+		2	2	8		
	2	8	8	8		
	thousands	hundreds	tens	ones		

a)

hundreds	tens	ones		tens	ones
2 2	$3_{5\ 4}$	8	·	7	6
		2	6	6	0
	+		2	2	8
		2	8	8	8
		thousands	hundreds	tens	ones

b)

Figure 2: Slovenian standard multiplication algorithm.

We first multiply the multiplier by the highest digit of the multiplicand in two-digit multiplication with tens. Since we have multiplied with tens, we add a zero at the end of the partial product. If we have multi-digit multiplicands, we add as many zeros as the value of the most significant digit. Writing down zeros helps students to understand the final addition of partial products. We continue with the remaining digits of the multiplicand. If a partial product of digits (e.g. 7 8) results in a number that is higher than 10, ones of the partial product (6 in 56) are written off, tens of the partial product are transferred to the next place value digit. In the next step, these tens are added to the new partial product (7 3+1). In the end, we sum up two partial products. When the students understand the algorithm by reference to place values, we start by dropping the addition of zeros at the end. The algorithm takes the classic staircase shape (Figure 3, left). The staircase shape is sometimes taught directly, producing the so-called "lining up procedure" -multiply, move to the right, multiply, add. The algorithm in Slovenian schools is often illustrated with an array field (Figure 3, right).

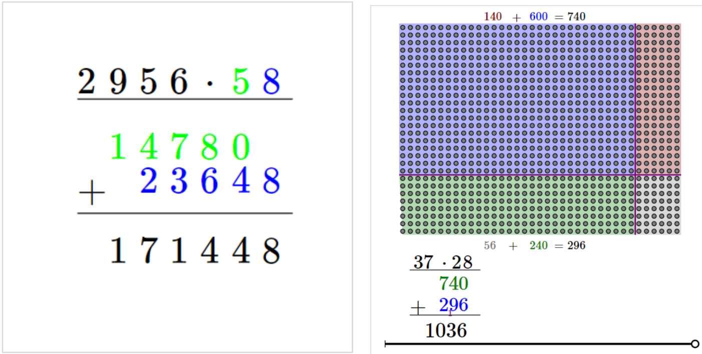


Figure 3: The »Staircase« shape of the Slovenian standard algorithm of multiplication and array model for illustration of it.

Reprinted from Bajramovič et al., 2014, p. 264, 261.

There are several "neuralgic" points for understanding this procedure. We have already mentioned the issue of "dropping" zeros. Similar is the dilemma under which of the two factors (multiplicand or multiplier) the partial product should be written. This is an important question for students who do not understand the procedure. The procedure shown in Figure 1b is mainly used in fourth and fifth grade, but since subject teachers in later grades often "drop the line" (see Figure 5), teachers in higher grades more often write the product below the multiplicand. This can be confusing for the students if it is shown without explaining the rationale.

The standard algorithm shown in Figure 2 is used in several other countries with slight changes (e.g. the spatial arrangement of the factors could be vertical instead of horizontal, see Figure 4.

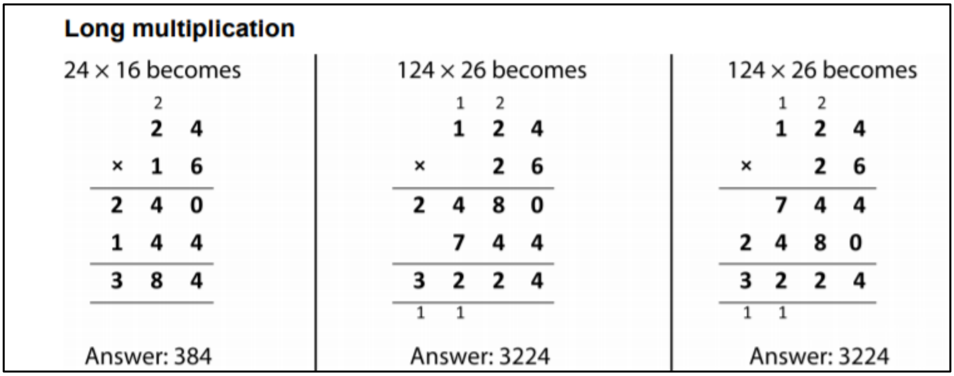


Figure 4: UK and Wales standard algorithm of multiplication.
Reprinted from Department for Education (2013, p.143)

One of the biggest advantages of a standard algorithm is its generality. It always works, the procedure is always the same, no matter what numbers we multiply, no matter what digits these numbers have, no matter how many digits we have. Ma (1999) stressed the importance of two digit by two-digit multiplication. Ma compared the lessons of Chinese and American teachers. The results show that more Chinese than American teachers are aware that two-digit multiplication occupies a particularly important place among multiplication algorithms. The multiplication of two-digit numbers is a central concept in the multiplication of multi-digit numbers. If students understand this multiplication, they will also understand other multiplication algorithms (e.g. multiplication of three-digit numbers).

In general, it is possible to figure out how to multiply any two numbers without the standard algorithm, but the strategy cannot always be generalized. Using the standard multiplication algorithm, we solve the problem of multiplication for all cases once and for all.

Another advantage is time efficiency. The record itself could be further optimized. We start by dropping zeros at the end of the partial products. Then we omit the first partial product, if the case, when the number of tens of the multiplier equals 1 (see figure 5b). Over time, we also discard small numbers that indicate how much higher place values we have gained when swapping with lower place values (e.g. 4 hundreds for 40 tens). We therefore only record what is really necessary (see Figure 5c and Figure 2a). With this, we further develop the speed of execution.

$$\begin{array}{rcccccc} & 3_4 & 4_4 & 5 & \cdot & 1 & 9 \\ \hline & 3 & 4 & 5 & & & \\ + & 3 & 1 & 0 & 5 & & \\ \hline & 6 & 5 & 6 & 5 & & \end{array}$$

a)

$$\begin{array}{cccccc} 3_4 & 4_4 & 5 & \cdot & 1 & 9 \\ 3 & 1 & 0 & 5 & & \\ \hline 6 & 5 & 6 & 5 & & \end{array}$$

b)

$$\begin{array}{cccccc} 3 & 4 & 5 & \cdot & 1 & 9 \\ 3 & 1 & 0 & 5 & & \\ \hline 6 & 5 & 6 & 5 & & \end{array}$$

c)

Figure 5: Dropping the line when the highest place value of the multiplier equals 1.

A multi-digit multiplication algorithm is built up from single-digit operations using the place value system and the basic properties of numbers such as distributivity. The general operations are reduced to the single-digit number facts. Regardless of students' level of understanding, students without instant recall of these foundational single-digit number facts are severely handicapped in their attempts to reach the next level of mathematics.

Several authors (e.g. Hickendorff et al., 2019; Young-Loveridge & Mills, 2009) reported that the standard algorithm for multiplication could be problematic for students. Van de Walle and colleagues (2014) pointed out that the numbers are viewed as single digits and not as decimal units. Only when adding together partial products we pay attention to how many digits the end product should have. Van de Walle and colleagues (2014) highlighted that it is very important to emphasize the importance of the digits' place value to reduce the risk of errors. When explaining the algorithm, the teacher should also create a good graphical representation of the algorithm on the blackboard (Lampert, 1986). This will help the students to better memorize spoken explanations. The graphical representation of the algorithm should be clear and transparent; the steps of the algorithm must be clearly visible.

Ma (1999) gave a more refined discussion of why rote learning might take place in the context of multi-digit multiplication: This is the case when the teacher does not possess a deep understanding of the underlying mathematics to explain it well. Teaching multi-digit multiplication using procedural methods does not give the student a proper understanding of place value and the distributive property. 70 % of teachers in the United States stated that the problem was an incorrect procedure lining up, while 30 % concluded that students did not understand the rationale of the algorithm. The teachers agreed that there is a problem with the learning comprehension for the students, which is a direct reflection of the teachers' teaching methods. Even though teachers have difficulties teaching multi-digit multiplication and notice similar errors every year when the algorithm is taught, these teachers do not seem to take steps to change teaching methods. The "carrying out the lining up" algorithm is taught with a procedurally directed method, which refers to the term "place-value" as the location of the numbers. The procedurally directed approach "verbalized the algorithm so it can be carried out correctly" yet by doing these, teachers are not providing the understanding of the importance of the definition of true place value (Ma, 1999, p. 29). Although teachers used other methods like using lined paper or a grid to position the "zeros" in the placeholder, the teacher merely suggested placing the numbers correctly. The term "place-value" was not introduced to students as a mathematical concept, but as labels for columns where they should put numbers" (Ma, 1999).

Alternative Algorithms

The advantage of working with non-standard or alternative algorithms is emphasized by many authors (e.g. Ambrose et al., 2003; Van De Walle et al., 2014). Focusing on empirically based studies, Randolph and Sherman (2001, p. 484) stated that "alternative algorithms offer a vehicle for a deeper understanding of mathematics". Fuson (2003) argued that various alternative algorithms could be suitable for multi-digit multiplication. Each of the alternatives has pros and cons and it is the teachers' job to study those pros and cons to choose the alternative algorithm. West (2011) listed nine alternative algorithms. We present three algorithms we found suitable for the fifth grade (students aged 10-11). We have to note that all the presented alternative algorithms are actually based on the same concepts as the standard algorithm, namely distributivity property $a \cdot (b + c) = a \cdot b + a \cdot c$ when decomposing multiplicands to decimal units. Alternative algorithms only represent the distributive property with a different model.

Area multiplication algorithm

The area multiplication algorithm uses "multiple representations to explain the multiplication process and can help students make connections to algebra and algebraic thinking" (West, 2011, p. 3). West (2011) presented the multiplication of $14 \cdot 12$ with an area model. First, we draw a rectangle with a height of 12 and width of 14, as we can see in Figure 6. The next step in the use of an area algorithm is to decompose the multiplicator and the multiplicand to tens and ones. Each summand is written up on one column or on the side of a row. In each sub-area, we calculate partial products of numbers that are entitled to certain sub-area. After calculating partial products, we sum them up.

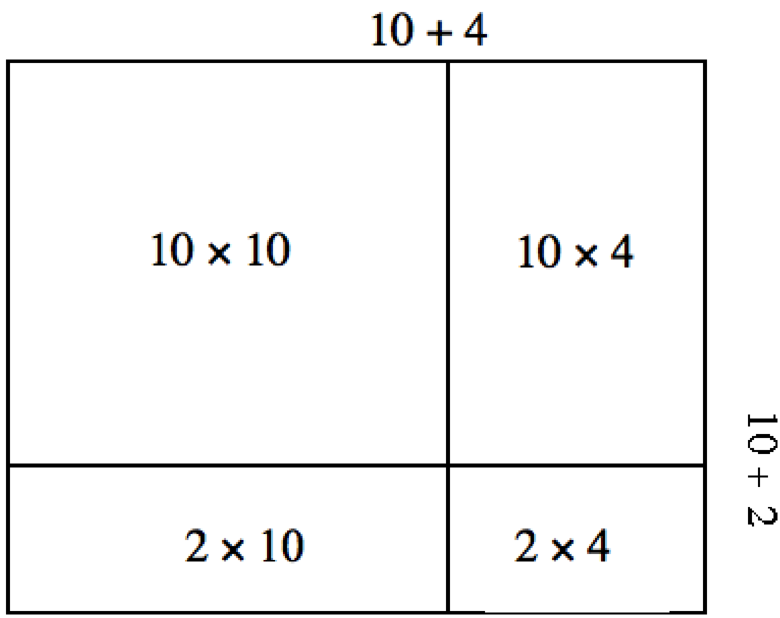


Figure 6: Area algorithm. Grid method.
Reprinted from West, 2011, p. 4.

West (2011) indicated that the area algorithm helps students to establish a fundamental understanding of a variety of basic math concepts. It can be used for calculations or only as a tool for a conceptual explanation of the standard algorithm. West (2011) also highlighted the illustration of commutative property of multiplication that can be illustrated with the area algorithm. Very similar to the area algorithm is the array-based algorithm introduced by Young-Loweridge and Mills (2009), depicted in Figure 7.

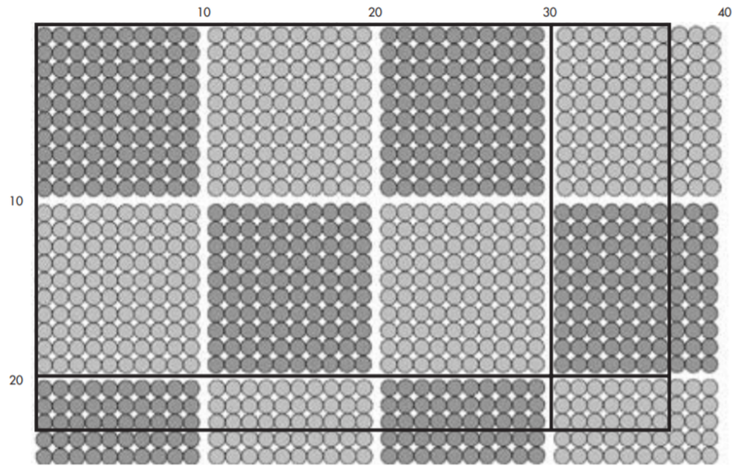


Figure 7: Area algorithm. Array model.
Reprinted from Young-Loweridge, & Mills, 2009, p. 53).

In England and Wales, the area algorithm is known as The grid method (or box method) and is often taught to pupils in primary or elementary school. It has been a standard part of the national primary school mathematics curriculum in England and Wales since the late 1990s.

Lattice multiplication algorithm

A lattice multiplication algorithm is "algorithmically identical to the traditional long multiplication method but breaks the process into smaller steps" (West, 2011). Figure 8 shows what the algorithm would look like if one wanted to multiply 453 · 25.

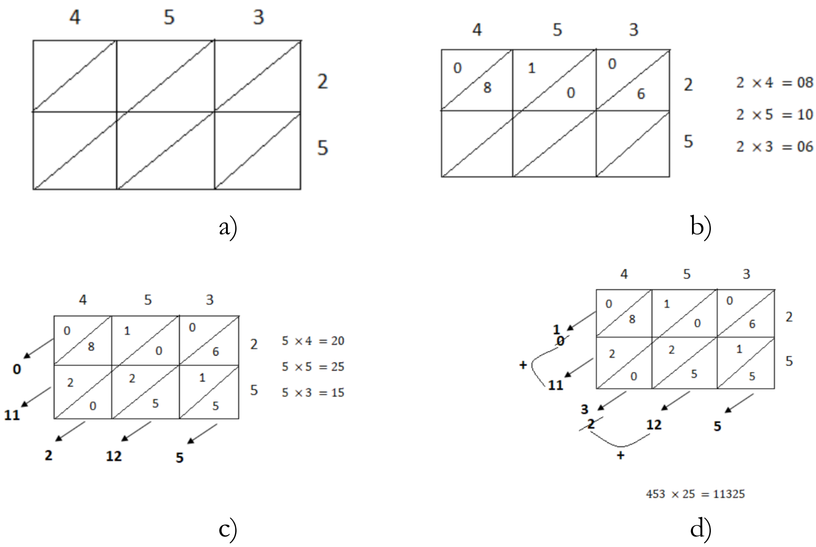


Figure 8: Lattice multiplication algorithm.
Reprinted from West, 2011, p. 5.

The area algorithm exposes decimal units of each digit in the number, lattice multiplication algorithm does not – in the first place. You still need a rectangle divided into as many columns as there are digits in the multiplier, and as many rows as there are digits in the multiplicand. We write multipliers across the top and down the right side, lining up the digits with the squares. (Figure 5a). In the case of $453 \cdot 25$, we obtain two rows, in each row, there are three squares (left, middle and right) and in each square, there are two triangles (upper and bottom). Two triangles are meant for each digit in partial products -the upper left triangle for tens and the lower right triangle for ones of the product. If the product does not have tens, then we write a zero in the upper left triangle (Figure 5b). There are two triangles for a reason -we know partial products will only have two or fewer digits.

Let us now consider where the lattice ones of the product $453 \cdot 25 = 11325$ are located. Ones in 11325 are obtained by multiplying ones of the multiplier with ones of the multiplicand $5 \cdot 3 = 15$. Therefore, ones are located in the bottom row, in the right square and in the bottom triangle (Figure 8c). Tens of the product can be obtained in three ways (Figure 8c): (a) multiplying ones of the multiplicand by tens of the multiplier (upper row, right square, bottom triangle), (b) multiplying tens of the multiplicand by ones of the multiplier (bottom row, middle square, bottom triangle) or (c) by carrying over the ones obtained in the product of multiplicand ones by multiplier ones (bottom row, right square, upper triangle). Hundreds of the product can be obtained (Figure 8c): (a) by multiplying tens of the multiplicand by tens of the multiplier (upper row, middle square, upper triangle), (b) by multiplying hundreds of the multiplicand by ones of the multiplier (bottom row, left square, upper triangle), (c) by carrying over the ones obtained in a product of multiplicand ones by multiplier tens (upper row, right square, upper triangle), or (d) by carrying over the ones obtained in a product of multiplicand tens by multiplier ones (upper row, middle square, upper triangle). Graphically that can be depicted as adding together the numbers along the diagonals (Figure 8c). If we get two-digit sums, we need to carry them to the next place and then record the final answer of multiplication (Figure 8d).

Line multiplication algorithm

Another algorithm used as an alternative is called the line multiplication algorithm. It is also a graphic representation of multiplication. We draw as many sets of vertical lines, as there are digits in the multiplier and as many sets of horizontal lines, as there are digits in the multiplicand. One set of lines represents the size of the number. For instance, if we want to multiply 22 by 13, we will draw lines like in Figure 9.

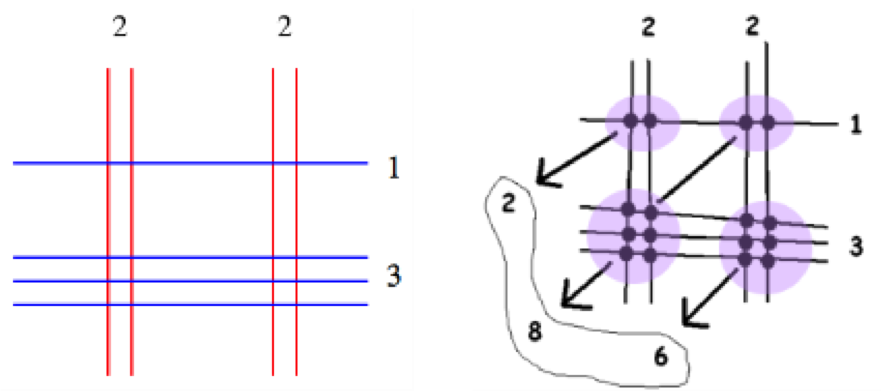


Figure 9: Line algorithm.
Reprinted from West, 2011, p. 6-7.

The next step is to highlight the intersecting points. To find the product, we count the intersecting points in each highlighted set and add diagonally. Just like with lattice multiplication, when adding diagonally we can get two-digit numbers, which means we must regroup the numbers and carry on tens to the next place (West, 2011).

The method works, because the number of parallel lines is like decimal placeholders and the number of dots at each intersection is a product of the number of lines. You are then summing up all the products that are coefficients of the same power of 10. Thus in the example shown in Figure 9: $22 \cdot 13 = (2 \cdot 10 + 2) \cdot (1 \cdot 10 + 3) = 2 \cdot 1 \cdot 10^2 + [2 \cdot 1 \cdot 10 + 3 \cdot 2 \cdot 10] + 2 \cdot 3 = 286$.

Comparing alternative algorithms

For the area algorithm of multiplication, Randolph and Sherman (2001) suggested that it improves the understanding of decimal units in multiplying two-digit numbers. You do not need any regrouping; you just multiply numbers with each other. This algorithm represent a sketch -multiplying this way helps the student to rest their brain, and it is fast and easy to calculate the product. Fuson (2003) indicated that the area algorithm of multiplication means an easier way of multiplying because it is gradual. With the lattice algorithm the teacher can identify multiplication facts for which students consistently find incorrect products. This model is divided into three main steps, which helps the student be organized and not get confused. Another benefit of the lattice algorithm is its appearance which students find appealing.

There are, of course, various disadvantages of alternative algorithms. All of them are more time consuming since you need to draw an array or a lattice or sets of parallel lines. The line multiplication algorithm is hard to use when digits are bigger since the picture becomes blurred, the problem occurs also when you multiply three-digit numbers by three-digit numbers.

Aim of the Study

While we studied different alternative algorithms, we noticed that many authors (Fuson, 2003; Jazby & Pearn, 2015; Van de Walle, 2005; West, 2011) indicate the problematics of the standard algorithm of written multiplication. The model is not easy to understand, and its use can lead to several errors, because of a lack of understanding (Leung, 2006). Therefore, we wanted to examine if any of the alternative algorithms can benefit fifth graders.

Method

A pedagogical experiment was conducted in order to answer the following question: Can teaching alternative algorithms contribute to students' two-digit multiplication fluency?

Design of the study

Students were first introduced to a standard algorithm of the two-digit multiplication. After that, some changes were applied in experimental groups. The experimental group EG1 was additionally introduced to an area algorithm, and the experimental group EG2 was additionally introduced to three alternative algorithms: an area algorithm, a lattice algorithm, and a line algorithm. During the experiment students in experimental groups did not use only the standard algorithm of two-digit multiplication, they were encouraged to use some of the alternative algorithms. The use of alternative algorithms was not mandatory, alternatives were introduced as a simple way and as a help when the standard algorithm might be too difficult to use. The experiment lasted one month (April–May 2018). After a month, we checked the participants' knowledge. Students had 45 minutes to solve the final test. On the final test, students themselves chose with which algorithm they would calculate.

Sample

We included a sample of 73 students of the fifth grade from two public elementary schools, where 55 % of students were boys. Students were divided into three groups, one control (CG) and two experimental groups (Table 1). Groups were formed according to pre-existing classes.

Table 1: Sample structure

	f	*f* %
Control group CG	24	33
Experimental group 1 -EG1	24	33
Experimental group 2 -EG1	25	34
together	73	100

Instrument

Data were collected with two different tests. To ensure the validity of the pedagogical experiment, we used TIMSS 2011 tasks for the fourth grade as initial testing. Examples of the TIMSS 2011 released tasks are in Figure 10. In task 28 students had to find the result of the offered products that is closest to the given product. In task 32, students needed to draw a bisector of the two-dimensional figure. Students' job in task 37 was to find the fraction that is bigger than $\frac{1}{2}$.

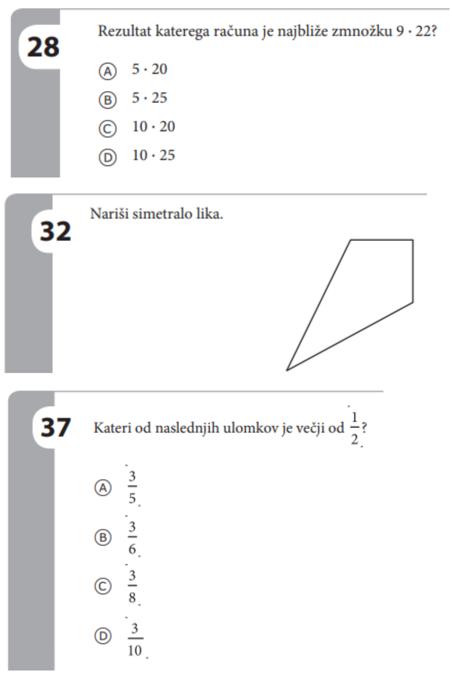


Figure 10: Some of the TIMSS 2011 tasks.

The instrument for final testing was designed for the purposes of the study and included ten numerical expressions: $12 \cdot 53$, $24 \cdot 12$, $44 \cdot 33$, $67 \cdot 47$, $27 \cdot 35$, $43 \cdot 18$, $58 \cdot 14$, $27 \cdot 89$, $94 \cdot 29$, $72 \cdot 68$. Numerical expressions were carefully chosen to represent two-digit multiplication.

Data analysis

The collected data was statistically processed in the program IDM SPSS 22. In the first part of the study, we used statistical inference to make predictions possible, and in the second part of the research, we used the Kruskal–Wallis H test to check for differences between EG1, EG2, and CG.

Results

TIMSS assignments cover different areas of mathematics and are thoughtfully created to cover different taxonomic levels. This allows us to evaluate students' previous knowledge. The results of the initial test (TIMSS) and the final test are presented in Table 2.

Table 2: Results of the initial test

		initial test results (%)	
	N	Mean (%)	Std. Dev. (%)
EG1	24	78.6	22.8
EG2	25	79.1	17.5
CG	24	82.1	20.7

Results in Table 1 show that tested groups performed similarly in both tests; inferential statistics agrees (Kruskal–Wallis $H = 0.732$, $df = 2$, $p = .693$).

The correctness of the products

Within the framework of the pedagogical experiment, alternative algorithms were adhered in the experimental group. Table 3 displays the results of the final test, regarding only correctness.

Table 3: Results of the final test

		final test results (%)	
	N	Mean (%)	Std. Dev. (%)
EG1	24	76.7	25.0
EG2	25	76.0	21.8
CG	24	80.8	25.5

We can see that the control group performed a little better in the final test as well. After introducing one alternative algorithm in EG1 and three alternative algorithms in EG2, statistical differences among students in different groups could not be confirmed (Kruskal–Wallis H = 1.582, df = 2, p = .453). We conclude that introducing alternative algorithms did not harm the correctness of the computational results.

Multiplication number sentences used in the final test were of different difficulties. Students' success with different number sentences is presented in Table 4. The order of number sentences is determined by the decline in CG performance.

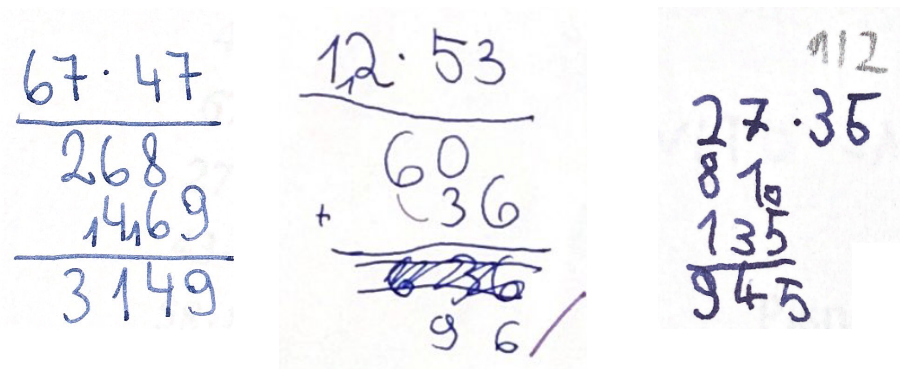
Table 4: Results regarding specific number sentences

Number sentence	EG1		EG2		CG	
	f	*f*%	*f*	*f*%	*f*	*f*%
24 · 12	21	88	23	92	21	88
94 · 29	16	67	18	72	21	88
27 · 35	20	83	18	72	21	88
43 · 18	17	70	22	88	20	83
44 · 33	19	79	16	64	20	83
12 · 35	21	88	24	96	19	79
67 · 47	18	75	16	64	19	79
72 · 68	18	75	17	68	18	75
27 · 89	16	67	15	60	15	63

The differences between groups are minimal. In some places, the difficulty of number sentences changed as students in different groups perceived it. Number sentence $12 \cdot 35$ is one such example. The number sentence is quite easy since all digits are small. The CG groups showed only 79 % success, in both experimental groups the success was much higher (88 % and 96 %). The number sentence $94 \cdot 29$ shows the opposite characteristics. In this number sentence, we have two nines as digits. The transition between places occurs when multiplying ones of the multiplicand with ones of the multiplier. The result was expected to be lower, which happened in both experimental groups (67 % and 72 %); however, the result in the control group was relatively high (88 %).

Students' strategies

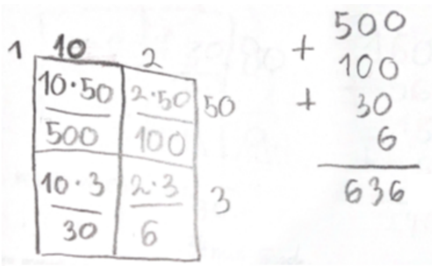
Using a standard multiplication algorithm, the students in the CG group wrote each digit of the same decimal unit one underneath the other. The majority of students did not write down the digit zero while multiplying the tens of the multiplier by the multiplier (Figure 11a). Students who used the standard algorithm in an improper manner often encountered a problem that was due to a misunderstanding of the space value. One of the examples is shown in Figure 11b. When adding together the partial products, no digit 0 was assigned to the first partial product, so the result was only shown as a two-digit. The record in Figure 11c enumeration of the digits in the multiplicand serves as help in remembering the order of the partial products. We can also observe a small digit zero.



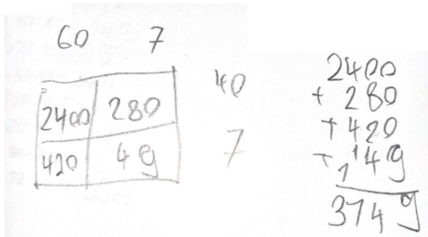
a) b) c)

Figure 11: Students' records of the standard algorithm of multiplication

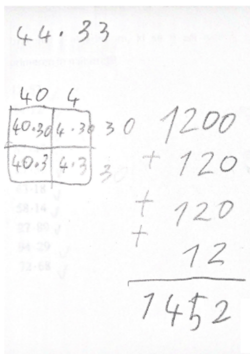
We observed at least three different strategies for the area multiplication algorithm. The first and most common is shown in Figure 12a. Students wrote down each partial calculation that belongs to a particular square. When they had all the products, they totalled the products with a written sum calculation on the side. The other strategies for using the area multiplication algorithm consisted of recording only partial products (Figure 12b) or writing down only partial number sentences without products (Figure 12c).



a)



b)



c)

Figure 12: Students' records of the area multiplication algorithm

The lattice multiplication algorithm was not used often. Using a lattice multiplication algorithm, the students first calculated partial products and then summed them. Students used arrows to depict transferring the digits to the larger decimal unit. As it can be seen in Figure 13, a student drew an arrow and wrote down which digit transfers to a larger decimal unit. This strategy helped the student to calculate the final product.

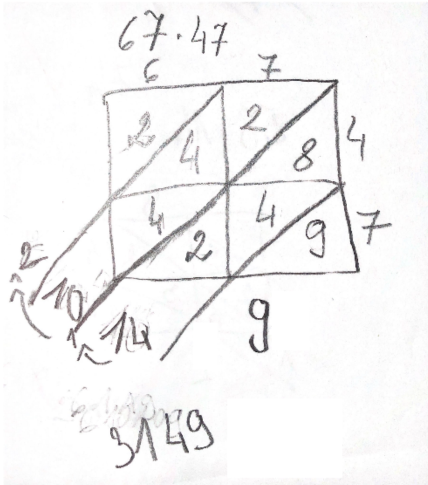


Figure 13: Students' record of the lattice multiplication algorithm

Time efficiency

Computational fluency has at least two dimensions: Correctness (of the result) and time efficiency of the calculation. Table 5 shows the average time (in minutes) taken to complete the final test, which included ten number sentences listed above.

Table 5: Time efficiency

	N	Mean (minutes)	Std. Dev.(minutes)
area	41	19.57	1.3
lattice	8	18.33	1.7
standard	24	9.23	0.5

We see that the time efficiency of the algorithm implementation is by far the best in the standard algorithm. We also noticed that the standard deviation is the lowest, which means that the standard algorithm is about equally effective for different types of learners. The differences were statistically significant (Kruskal -Wallis $H = 39.007, p = .000$) in favor of the standard algorithm.

Students from experimental groups EG1 and EG2 had an option to decide which algorithm to use on the final test – standard or any of the alternative ones. The results of student choices are presented in Table 6.

Table 6: Students who chose standard or one of the alternative algorithms.

		area		lattice		line		standard	
	N	*f*	*f* %	*f*	*f* %	*f*	*f* %	*f*	*f* %
EG1	24	24	100	0	0	0	0	0	0
EG2	25	17	71	8	29	0	0	0	0
CG	24	0	0	0	0	0	0	24	100

The results in Table 6 show that all students in EG1 and EG2 have chosen alternative algorithms. In EG2 they had four algorithms to choose from (one standard and three alternatives). The majority (71 %) chose the area algorithm, the rest chose the lattice algorithm.

Discussion and Conclusion

Kadum (2005) emphasizes the importance of understanding algorithms used by students. By deciding to introduce alternative multiplication algorithms into the learning process, we wanted to bring students closer to the understanding of the two-digit multiplication algorithm. All but three students have used algorithms in an appropriate way. Iljič (2017) exposed that students use the correct interpretation of the algorithm if their steps and understanding of the concept of the algorithm are clear and organized. The problem of the "missing zero" has also been addressed in several other studies to point out shortcomings of the standard algorithm (e.g. Hickendorff et al., 2019; Young-Loveridge & Mills, 2009). Norton (2012) pointed out that many of the students' mistakes were due to a poor understanding of the algorithms, which led to the algorithms being confused with each other (in this case the algorithm for addition and algorithm for multiplication).

There were no significant differences between the students in the control group and experimental groups in the average score in the test of written multiplication. Even in the more difficult calculations of the two-digit multiplication (comp. Table 4) the differences were not statistically significant. Students in the control group were almost three times faster in completing the final test of written multiplication. This suggests that using alternative algorithms takes more time for students than using a standard multiplication algorithm. Most students in the control group needed between 10 and 20 minutes to complete the final test. All students in the control group completed the test in less than 15 minutes.

The control group performed the final test immediately after two weeks of learning and consolidating the standard algorithm, and the experimental groups performed the final test one month after the introduction of alternative algorithms. Students were not advised which methods of written multiplication they should use. They have decided themselves whether to use the traditional algorithm or any of the alternative methods. This delay is considered as one of the limitations of the study since retention was measured only in experimental groups.

The results of our study show that the introduction of alternative algorithms does not affect the correctness in computing. The results are somehow inconsistent with some other studies (e.g. Fuson, 2003; West, 2001) which claim that the introduction of alternative algorithms is beneficial for students. It seems that the advantages lie in the conceptual part, whereas the procedural part remains unchanged.

None of the students from the experimental groups chose a standard algorithm, all of them used one of the alternative algorithms. The results indicate that students prefer alternative algorithms over the standard algorithm. Similarly, Iljič (2017) conducted research among students at a faculty, where she investigated the use of alternative algorithms of multiplication. The students were more interested in using alternative algorithms, which also led to a better understanding of the algorithm.

The area algorithm was chosen by 41 of 49 students. The results are in line with several other study findings. The area algorithm allows a process of multi-digit multiplication to be represented as a rectangle with the sides corresponding to the two factors, and this is consistent with Davis' view that "the most flexible and robust interpretation of multiplication is based on a rectangle" (2008, p. 88). Also, Young-Loweridge and Mills report in their work with 46 students (11–13 years) that the adoption of arrays representing the area (comp. Figure 3b) can be useful to improve the students' understanding of multi-digit multiplication. The students' preference for the area algorithm over the standard algorithm was also reported by Bobis (2007). In addition, Jazby and Pearn (2015) report results indicating that the use of the area multiplication algorithm is the most effective tool for explaining the standard algorithm. However, the use of alternative algorithms slowed down students in comparison to using the standard algorithm, and the average computing time was about three times shorter.

One of the basic didactic principles of arithmetic algorithms is the "delay" principle (Van de Walle et al., 2014). The first algorithm that students usually learn is the standard addition algorithm. Some students may learn standard algorithms from older relatives. However, it is highly unlikely that they will invent them themselves. Standard algorithms are therefore usually implemented by the teacher. It is also the teacher's task to make the algorithms understandable to all students. The use of algorithmic procedures in arithmetic slows down the development of the number sense, so teachers are advised to wait with the introduction of standard algorithms at least until students are able to add up to 100 fluently. When students master the standard algorithms, they quickly determine its effectiveness and use it even in situations where there is no necessity to use them, for example 999 + 1.

The results of the presented study show that students who use alternative algorithms do not achieve better computational results in multiplication than students who use a standard algorithm. However, the results also show that students prefer alternative algorithms over the standard algorithm. The area multiplication algorithm especially stands out. We agree with Van de Walle et al. (2014) that the array (also area) model promotes a visual demonstration of the commutative and distributive properties and that it can also be linked to successful representations of the standard algorithm for multiplication. The array representation of the multiplication algorithm is already present in Slovenian school practice (comp. Figure 7b). Clivaz (2017) pointed out that it is important that teachers understand the algorithm well. From our own experience, we must stress that it is very important that the teacher takes enough time to explain the algorithm to the students. It takes a little longer to explain the individual steps and their meaning, but this will reduce the time spent on consolidating the algorithm.

The importance of teaching computational algorithms, or at least the amount of time spent teaching them at school, has been frequently questioned over the past decade (Fuson, 2003). In modern society, calculations are made by technology, but man is needed to solve problems. We believe that the introduction of alternative algorithms opens up new ways to compromise between "traditional school content" represented by standard algorithms and the demand for the development of problem-solving skills represented by alternative algorithms.

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